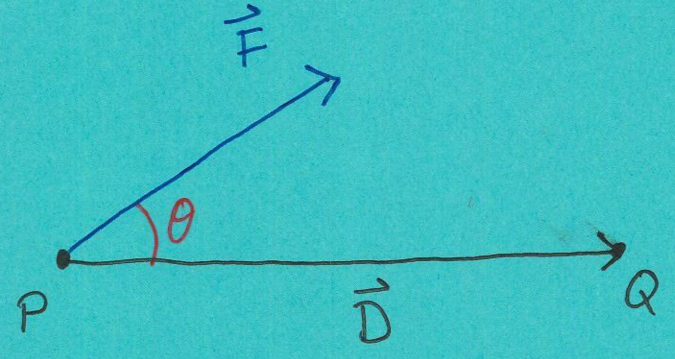
Lecture 4

Application of dot product: Work

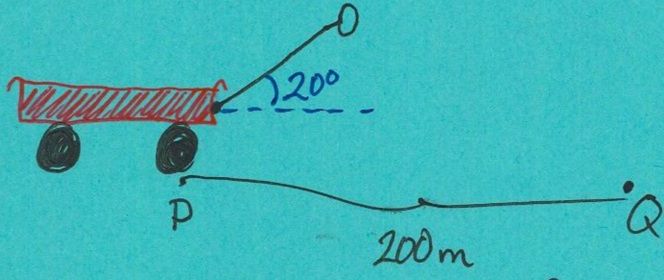
Suppose we have a constant force $\vec{F}$, which moves an object from point P to point Q. Let $\vec{D}=\vec{PQ}$ be the displacement vector. Then we have:



The work done on the object by $\vec{F}$ is the component of $\vec{F}$ in the direction of the displacement, times the displacement distance, i.e.,

$$\text{Work} = \boxed{W = (|\vec{F}|\cos\theta)\,|\vec{D}| = \vec{F}\cdot\vec{D}}$$

Ex: A child pulls a red wagon a distance of 200 m by exerting a force of 100 N on the handle which makes an angle of 20° with the horizontal.



How much work has the child done in moving the wagon?

Sol: $W = (|\vec{F}| \cos\theta)|\vec{D}| = (100 \cos 20°) \cdot 200 = 20000 \cos 20°$

≈ 18793.85 J

Ex: A particle is moved from $P = (3,2,4)$ to $Q = (1,6,7)$ by the force $\vec{F} = \langle 1, 3, 2 \rangle$. How much work has $\vec{F}$ done on the particle?

Sol: $W = \vec{F} \cdot \vec{PQ} = \langle 1, 3, 2 \rangle \cdot \langle -2, 4, 3 \rangle = -2 + 12 + 6 = 16$

12.4 - The Cross Product

Given two nonzero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$, we would like a third vector, call it $\vec{w} = \langle w_1, w_2, w_3 \rangle$, which is orthogonal to both $\vec{u}$ & $\vec{v}$. So, we should have:

$$\vec{u} \cdot \vec{w} = u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \quad ①$$

&

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \quad ②$$

Take v_3 ① $- u_3$ ②:

$$\underbrace{(u_1 v_3 - u_3 v_1)}_{p} w_1 + \underbrace{(u_2 v_3 - u_3 v_2)}_{q} w_2 = 0 \quad ③$$

Taking $w_2 = -p$ and $w_1 = q$ we have a solution of ③

Plugging this back into ① & ② we can solve for w_3. So,

$$w_1 = u_2 v_3 - u_3 v_2, \quad w_2 = u_3 v_1 - u_1 v_3, \quad w_3 = u_1 v_2 - u_2 v_1$$

We call $\vec{w}$ the <u>cross product</u> of $\vec{u}$ and $\vec{v}$ and denote it $\vec{u} \times \vec{v}$. Since $\vec{u} \times \vec{v}$ is again a vector, we often call it the <u>vector product</u>.

Recall: determinants

The determinant of a:

1) 2×2 matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad} - \underline{bc}$$

$-$ $+$

2) 3×3 matrix is:

$$\begin{vmatrix} \overset{+}{a_1} & \overset{-}{a_2} & \overset{+}{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

An easy way to compute the cross product is with determinants: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\boxed{\begin{aligned} \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \hat{i}(u_2v_3 - u_3v_2) - \hat{j}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1) \\ &= \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle \end{aligned}}$$

Example: Find $\vec{u} \times \vec{v}$ if $\vec{u} = \hat{i} - 2\hat{j} + \hat{k}$ & $\vec{v} = 3\hat{i} + \hat{j} - 2\hat{k}$

Sol:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = [(-2)(-2) - (1)(1)]\hat{i} - [(1)(-2) - (1)(3)]\hat{j} + [(1)(1) - (-2)(3)]\hat{k}$$

$$= (4-1)\hat{i} - (-2-3)\hat{j} + (1+6)\hat{k} = 3\hat{i} + 5\hat{j} + 7\hat{k}$$

What about $\vec{v} \times \vec{u}$?

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \langle (1)(1) - (-2)(-2), -[(3)(1) - (-2)(1)], (3)(-2) - (1)(1) \rangle$$

$$= \langle 1-4, -(3+2), -6-1 \rangle = \langle -3, -5, -7 \rangle$$

$$= -(\vec{u} \times \vec{v})$$

The cross product takes in two <u>VECTORS</u> and outputs a <u>VECTOR</u>.

Properties of the Cross Product

Algebraic Properties

1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

2) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

3) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

4) $\vec{u} \times \vec{0} = \vec{0} = \vec{0} \times \vec{u}$

5) $\vec{u} \times \vec{u} = \vec{0}$

6) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

7) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

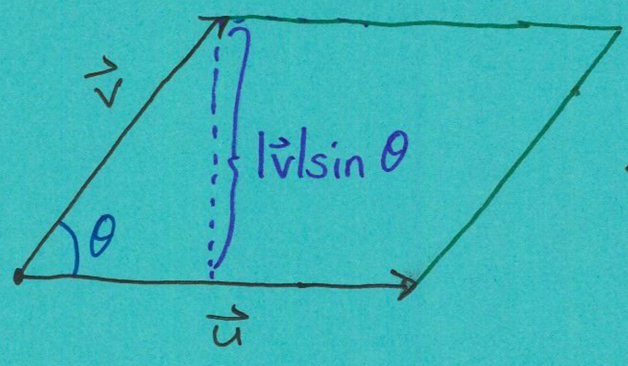
Geometric Properties

8) $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

9) $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ & $\vec{v}$ are scalar multiples of each other, i.e, they are parallel $\underset{(c \neq 0)}{}$ $(\vec{u} = c\vec{v})$

10) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$, where θ $(0 \le \theta \le \pi)$ is the angle between $\vec{u}$ & $\vec{v}$.

11) $|\vec{u} \times \vec{v}|$ is the area of the parallelogram spanned by $\vec{u}$ & $\vec{v}$



$\leftarrow$ parallelogram spanned by $\vec{u}$ & $\vec{v}$

The area of a parallelogram is (base)·(height) so

$$\boxed{A = |\vec{u}| \cdot |\vec{v}| \sin\theta = |\vec{u} \times \vec{v}|}$$

Ex: Consider the three points
$P = (5,2,0)$, $Q = (2,6,1)$, $R = (5,0,6)$.

a) Find a vector perpendicular to the plane containing these points.

b) Find the area of the triangle ΔPQR.

Sol: a) The plane contains the vectors:
$\vec{PQ} = \langle -3, 4, 1 \rangle$ & $\vec{PR} = \langle 0, -2, 6 \rangle$

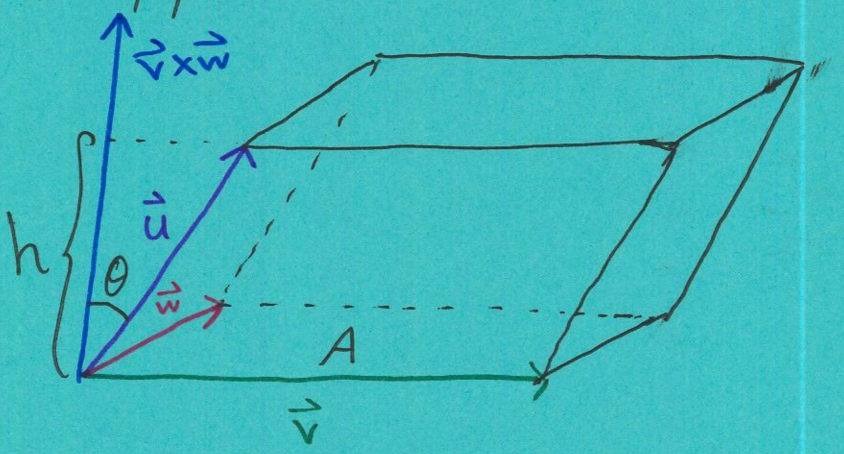
So a perpendicular vector is

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = \langle 24+2, -(-18-0), 6-0 \rangle = \langle 26, 18, 6 \rangle$$

b) The area of ΔPQR is half that of the parallelogram spanned by $\vec{PQ}$ & $\vec{PR}$. Thus the area of ΔPQR is

$$A = \frac{1}{2}\,|\vec{PQ}\times\vec{PR}| = \frac{1}{2}\sqrt{26^2+18^2+6^2} = \frac{1}{2}\sqrt{1036} \approx 16.09$$

Just as two vectors span a parallelogram, three vectors span a parallelpiped:



The ~~area~~ volume of a parallelpiped is (area of base)·(height)

So, $V = Ah$. We know $A = |\vec{v}\times\vec{w}|$ and

$h = |\vec{u}||\cos\theta|$ (we use $|\cos\theta|$ in case $\theta > \frac{\pi}{2}$).

So $V = A\,h = |\vec{v}\times\vec{w}|\,|\vec{u}|\,|\cos\theta| = |(\vec{v}\times\vec{w})\cdot\vec{u}| = |\vec{u}\cdot(\vec{v}\times\vec{w})|$

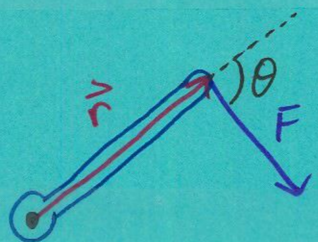
We call the expression $\vec{u}\cdot(\vec{v}\times\vec{w})$ the <u>scalar triple product</u> of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$, we can compute the scalar triple product as

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Fact: If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then the parallelpiped has zero volume, which means $\vec{u}, \vec{v}$, and $\vec{w}$ are coplanar (lie in the same plane).

Application: Torque



A force $\vec{F}$ acts at the end of a lever arm making an angle θ with the arm. The torque produced (at the pivot point) is the vector

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The magnitude of the torque vector is

$$|\vec{\tau}| = |\vec{r}||\vec{F}| \sin\theta.$$

Ex: A bolt is tightened by applying 50 N of force on the end of a .2 m wrench at an angle of 60°. Find the magnitude of the torque about the center of the bolt.

Sol:

$$|\tau| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin 60° = (.2)(50)\left(\frac{\sqrt{3}}{2}\right)$$

$$= 5\sqrt{3}$$